

CONSTRUCTION OF NORMAL NUMBERS VIA PSEUDO POLYNOMIAL PRIME SEQUENCES

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ABSTRACT. In the present paper we construct normal numbers in base q by concatenating q -ary expansions of pseudo polynomials evaluated at the primes. This extends a recent result by Tichy and the author.

1. INTRODUCTION

Let $q \geq 2$ be a positive integer. Then every real $\theta \in [0, 1)$ admits a unique expansion of the form

$$\theta = \sum_{k \geq 1} a_k q^{-k} \quad (a_k \in \{0, \dots, q-1\})$$

called the q -ary expansion. We denote by $\mathcal{N}(\theta, d_1 \cdots d_\ell, N)$ the number of occurrences of the block $d_1 \cdots d_\ell$ amongst the first N digits, *i.e.*

$$\mathcal{N}(\theta, d_1 \cdots d_\ell, N) := \#\{0 \leq i < n : a_{i+1} = d_1, \dots, a_{i+\ell} = d_\ell\}.$$

Then we call a number normal of order ℓ in base q if for each block of length ℓ the frequency of occurrences tends to $q^{-\ell}$. As a qualitative measure of the distance of a number from being normal we introduce for integers N and ℓ the discrepancy of θ by

$$\mathcal{R}_{N,\ell}(\theta) = \sup_{d_1 \dots d_\ell} \left| \frac{\mathcal{N}(\theta, d_1 \cdots d_\ell, N)}{N} - q^{-\ell} \right|,$$

where the supremum is over all blocks of length ℓ . Then a number θ is normal to base q if for each $\ell \geq 1$ we have that $\mathcal{R}_{N,\ell}(\theta) = o(1)$ for $N \rightarrow \infty$. Furthermore we call a number absolutely normal if it is normal in all bases $q \geq 2$.

Borel [2] used a slightly different, but equivalent (*cf.* Chapter 4 of [3]), definition of normality to show that almost all real numbers are normal with respect to the Lebesgue measure. Despite their omnipresence it is not known whether numbers such as $\log 2$, π , e or $\sqrt{2}$ are

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normal to any base. The first construction of a normal number is due to Champernowne [4] who showed that the number

$$0.1234567891011121314151617181920\dots$$

is normal in base 10.

The construction of Champernowne laid the base for a class of normal numbers which are of the form

$$\sigma_q = \sigma_q(f) = 0.\lfloor f(1) \rfloor_q \lfloor f(2) \rfloor_q \lfloor f(3) \rfloor_q \lfloor f(4) \rfloor_q \lfloor f(5) \rfloor_q \lfloor f(6) \rfloor_q \dots,$$

where $\lfloor \cdot \rfloor_q$ denotes the expansion in base q of the integer part. Davenport and Erdős [6] showed that $\sigma(f)$ is normal for f being a polynomial such that $f(\mathbb{N}) \subset \mathbb{N}$. This construction was extended by Schiffer [19] to polynomials with rational coefficients. Furthermore he showed that for these polynomials the discrepancy $\mathcal{R}_{N,\ell}(\sigma(f)) \ll (\log N)^{-1}$ and that this is best possible. These results were extended by Nakai and Shiokawa [17] to polynomials having real coefficients. Madritsch, Thuswaldner and Tichy [12] considered transcendental entire functions of bounded logarithmic order. Nakai and Shiokawa [16] used pseudo-polynomial functions, *i.e.* these are function of the form

$$(1.1) \quad f(x) = \alpha_0 x^{\beta_0} + \alpha_1 x^{\beta_1} + \dots + \alpha_d x^{\beta_d}$$

with $\alpha_0, \beta_0, \alpha_1, \beta_1, \dots, \alpha_d, \beta_d \in \mathbb{R}$, $\alpha_0 > 0$, $\beta_0 > \beta_1 > \dots > \beta_d > 0$ and at least one $\beta_i \notin \mathbb{Z}$. Since we often only need the leading term we write $\alpha = \alpha_0$ and $\beta = \beta_0$ for short. They were also able to show that the discrepancy is $\mathcal{O}((\log N)^{-1})$. We refer the interested reader to the books of Kuipers and Niederreiter [11], Drmota and Tichy [7] or Bugeaud [3] for a more complete account on the construction of normal numbers.

The present method of construction by concatenating function values is in strong connection with properties of q -additive functions. We call a function f strictly q -additive, if $f(0) = 0$ and the function operates only on the digits of the q -ary representation, *i.e.*,

$$f(n) = \sum_{h=0}^{\ell} f(d_h) \quad \text{for} \quad n = \sum_{h=0}^{\ell} d_h q^h.$$

A very simple example of a strictly q -additive function is the sum of digits function s_q , defined by

$$s_q(n) = \sum_{h=0}^{\ell} d_h \quad \text{for} \quad n = \sum_{h=0}^{\ell} d_h q^h.$$

Refining the methods of Nakai and Shiokawa [16] the author obtained the following result.

Theorem 1.1 ([14, Theorem 1.1]). *Let $q \geq 2$ be an integer and f be a strictly q -additive function. If p is a pseudo-polynomial as defined in (1.1), then there exists $\eta > 0$ such that*

$$\sum_{n \leq N} f(\lfloor p(n) \rfloor) = \mu_f N \log_q(p(N)) + NF(\log_q(p(N))) + \mathcal{O}(N^{1-\eta}),$$

where

$$\mu_f = \frac{1}{q} \sum_{d=0}^{q-1} f(d)$$

and F is a 1-periodic function depending only on f and p .

In the present paper, however, we are interested in a variant of $\sigma_q(f)$ involving primes. As a first example, Champernowne [4] conjectured and later Copeland and Erdős [5] proved that the number

$$0.2357111317192329313741434753596167\dots$$

is normal in base 10. Similar to the construction above we want to consider the number

$$\tau_q = \tau_q(f) = 0.\lfloor f(2) \rfloor_q \lfloor f(3) \rfloor_q \lfloor f(5) \rfloor_q \lfloor f(7) \rfloor_q \lfloor f(11) \rfloor_q \lfloor f(13) \rfloor_q \dots,$$

where the arguments of f run through the sequence of primes.

Then the paper of Copeland and Erdős corresponds to the function $f(x) = x$. Nakai and Shiokawa [18] showed that the discrepancy for polynomials having rational coefficients is $\mathcal{O}((\log N)^{-1})$. Furthermore Madritsch, Thuswaldner and Tichy [12] showed, that transcendental entire functions of bounded logarithmic order yield normal numbers. Finally in a recent paper Madritsch and Tichy [13] considered pseudo-polynomials of the special form αx^β with $\alpha > 0$, $\beta > 1$ and $\beta \notin \mathbb{Z}$.

The aim of the present paper is to extend this last construction to arbitrary pseudo-polynomials. Our first main result is the following

Theorem 1.2. *Let f be a pseudo-polynomial as in (1.1). Then*

$$\mathcal{R}_N(\tau_q(f)) \ll (\log N)^{-1}.$$

In our second main result we use the connection of this construction of normal numbers with the arithmetic mean of q -additive functions as described above. Known results are due to Shiokawa [20] and Madritsch and Tichy [13]. Similar results concerning the moments of the sum of digits function over primes have been established by Kátai [10].

Let $\pi(x)$ stand for the number of primes less than or equal to x . Then adapting these ideas to our method we obtain the following

Theorem 1.3. *Let f be a pseudo-polynomial as in (1.1). Then*

$$\sum_{p \leq P} s_q(\lfloor f(p) \rfloor) = \frac{q-1}{2} \pi(P) \log_q P^\beta + \mathcal{O}(\pi(P)),$$

where the sum runs over the primes and the implicit \mathcal{O} -constant may depend on q and β .

Remark 1.4. With simple modifications Theorem 1.3 can be extended to completely q -additive functions replacing s_q .

The proof of the two theorems is divided into four parts. In the following section we rewrite both statements in order to obtain as a common base the central theorem – Theorem 2.1. In Section 3 we start with the proof of this central theorem by using an indicator function and its Fourier series. These series contain exponential sums which we treat by different methods (with respect to the position in the expansion) in Section 4. Finally, in Section 5 we put the estimates together in order to proof the central theorem and therefore our two statements.

2. PRELIMINARIES

Throughout the rest p will always denote a prime. The implicit constant of \ll and \mathcal{O} may depend on the pseudo-polynomial f and on the parameter $\varepsilon > 0$. Furthermore we fix a block $d_1 \cdots d_\ell$ of length ℓ and N , the number of digits we consider.

In the first step we want to know in the expansion of which prime the N -th digit occurs. This can be seen as the translation from the digital world to the world of blocks. To this end let $\ell(m)$ denote the length of the q -ary expansion of an integer m . Then we define an integer P by

$$\sum_{p \leq P-1} \ell(\lfloor f(p) \rfloor) < N \leq \sum_{p \leq P} \ell(\lfloor f(p) \rfloor),$$

where the sum runs over all primes. Thus we get the following relation between N and P

$$\begin{aligned} N &= \sum_{p \leq P} \ell(\lfloor f(p) \rfloor) + \mathcal{O}(\pi(P)) + \mathcal{O}(\beta \log_q(P)) \\ (2.1) \quad &= \frac{\beta}{\log q} P + \mathcal{O}\left(\frac{P}{\log P}\right). \end{aligned}$$

Here we have used the prime number theorem in the form (cf. [21, Théorème 4.1])

$$(2.2) \quad \pi(x) = \text{Li } x + \mathcal{O}\left(\frac{x}{(\log x)^G}\right),$$

where G is an arbitrary positive constant and

$$\operatorname{Li} x = \int_2^x \frac{dt}{\log t}.$$

Now we show that we may neglect the occurrences of the block $d_1 \cdots d_\ell$ between two expansions. We write $\mathcal{N}(f(p))$ for the number of occurrences of this block in the q -ary expansion of $\lfloor f(p) \rfloor$. Then (2.1) implies that

$$(2.3) \quad \left| \mathcal{N}(\tau_q(f); d_1 \cdots d_\ell; N) - \sum_{p \leq P} \mathcal{N}(f(p)) \right| \ll \frac{N}{\log N}.$$

In the next step we use the polynomial-like behavior of f . In particular, we collect all the values having the same length of expansion. Let j_0 be a sufficiently large integer. Then for each integer $j \geq j_0$ there exists a P_j such that

$$q^{j-2} \leq f(P_j) < q^{j-1} \leq f(P_j + 1) < q^j$$

with

$$P_j \asymp q^{\frac{j}{\beta}}.$$

Furthermore we set J to be the greatest length of the q -ary expansions of $f(p)$ over the primes $p \leq P$, i.e.,

$$J := \max_{p \leq P} \ell(\lfloor f(p) \rfloor) = \log_q(f(P)) + \mathcal{O}(1) \asymp \log P.$$

Now we show that we may suppose that each expansion has the same length (which we reach by adding leading zeroes). For $P_{j-1} < p \leq P_j$ we may write $f(p)$ in q -ary expansion, i.e.,

$$(2.4) \quad f(p) = b_{j-1}q^{j-1} + b_{j-2}q^{j-2} + \cdots + b_1q + b_0 + b_{-1}q^{-1} + \cdots$$

Then we denote by $\mathcal{N}^*(f(p))$ the number of occurrences of the block $d_1 \cdots d_\ell$ in the string $0 \cdots 0b_{j-1}b_{j-2} \cdots b_1b_0$, where we filled up the expansion with leading zeroes such that it has length J . The error of doing so can be estimated by

$$\begin{aligned} 0 &\leq \sum_{p \leq P} \mathcal{N}^*(f(p)) - \sum_{p \leq P} \mathcal{N}(f(p)) \\ &\leq \sum_{j=j_0+1}^{J-1} (J-j) (\pi(P_{j+1}) - \pi(P_j)) + \mathcal{O}(1) \\ &\leq \sum_{j=j_0+2}^J \pi(P_j) + \mathcal{O}(1) \ll \sum_{j=j_0+2}^J \frac{q^{j/\beta}}{j} \ll \frac{P}{\log P} \ll \frac{N}{\log N}. \end{aligned}$$

In the following three sections we will estimate this sum of indicator functions \mathcal{N}^* in order to prove the following theorem.

Theorem 2.1. *Let f be a pseudo polynomial as in (1.1). Then*

$$(2.5) \quad \sum_{p \leq P} \mathcal{N}^*(\lfloor f(p) \rfloor) = q^{-\ell} \pi(P) \log_q P^\beta + \mathcal{O}\left(\frac{P}{\log P}\right)$$

Using this theorem we can simply deduce our two main results.

Proof of Theorem 1.2. We insert (2.5) into (2.3) and get the desired result. \square

Proof of Theorem 1.3. For this proof we have to rewrite the statement. In particular, we use that the sum of digits function counts the number of 1s, 2s, etc. and assigns weights to them, i.e.,

$$s_q(n) = \sum_{d=0}^{q-1} d \cdot \mathcal{N}(n; d).$$

Thus

$$\begin{aligned} \sum_{p \leq P} s_q(\lfloor p^\beta \rfloor) &= \sum_{p \leq P} \sum_{d=0}^{q-1} d \cdot \mathcal{N}(p^\beta) = \sum_{p \leq P} \sum_{d=0}^{q-1} d \cdot \mathcal{N}^*(p^\beta) + \mathcal{O}\left(\frac{P}{\log P}\right) \\ &= \frac{q-1}{2} \pi(P) \log_q(P^\beta) + \mathcal{O}\left(\frac{P}{\log P}\right) \end{aligned}$$

and the theorem follows. \square

In the following sections we will prove Theorem 2.1 in several steps. First we use the “method of little glasses” in order to approximate the indicator function by a Fourier series having smooth coefficients. Then we will apply different methods (depending on the position in the expansion) for the estimation of the exponential sums that appear in the Fourier series. Finally we put everything together and get the desired estimate.

3. PROOF OF THEOREM 2.1, PART I

We want to ease notation by splitting the pseudo-polynomial f into a polynomial and the rest. Then there exists a unique decomposition of the following form:

$$(3.1) \quad f(x) = g(x) + h(x)$$

where $h \in \mathbb{R}[X]$ is a polynomial of degree k (where we set $k = 0$ if h is the zero polynomial) and

$$g(x) = \sum_{j=1}^r \alpha_j x^{\theta_j}$$

with $r \geq 1$, $\alpha_r \neq 0$, α_j real, $0 < \theta_1 < \dots < \theta_r$ and $\theta_j \notin \mathbb{Z}$ for $1 \leq j \leq r$.

Let γ and ρ be two parameter which we will frequently use in the sequel. We suppose that

$$0 < \gamma < \rho < \min \left(\frac{1}{4(k+1)}, \frac{\theta_r}{2} \right).$$

The aim of this section is to calculate the Fourier transform of \mathcal{N}^* . In order to count the occurrences of the block $d_1 \dots d_\ell$ in the q -ary expansion of $\lfloor f(p) \rfloor$ ($2 \leq p \leq P$) we define the indicator function

$$\mathcal{I}(t) = \begin{cases} 1, & \text{if } \sum_{i=1}^{\ell} d_i q^{-i} \leq t - \lfloor t \rfloor < \sum_{i=1}^{\ell} d_i q^{-i} + q^{-\ell}; \\ 0, & \text{otherwise;} \end{cases}$$

which is a 1-periodic function. Indeed, we have

$$(3.2) \quad \mathcal{I}(q^{-j} f(p)) = 1 \iff d_1 \dots d_\ell = b_{j-1} \dots b_{j-\ell},$$

where $f(p)$ has an expansion as in (2.4). Thus we may write our block counting function as follows

$$(3.3) \quad \mathcal{N}^*(f(p)) = \sum_{j=\ell}^J \mathcal{I}(q^{-j} f(p)).$$

In the following we will use Vinogradov's "method of little glasses" (cf. [23]). We want to approximate \mathcal{I} from above and from below by two 1-periodic functions having small Fourier coefficients. To this end we will use the following

Lemma 3.1 ([23, Lemma 12]). *Let α, β, Δ be real numbers satisfying*

$$0 < \Delta < \frac{1}{2}, \quad \Delta \leq \beta - \alpha \leq 1 - \Delta.$$

Then there exists a periodic function $\psi(x)$ with period 1, satisfying

- (1) $\psi(x) = 1$ in the interval $\alpha + \frac{1}{2}\Delta \leq x \leq \beta - \frac{1}{2}\Delta$,
- (2) $\psi(x) = 0$ in the interval $\beta + \frac{1}{2}\Delta \leq x \leq 1 + \alpha - \frac{1}{2}\Delta$,
- (3) $0 \leq \psi(x) \leq 1$ in the remainder of the interval $\alpha - \frac{1}{2}\Delta \leq x \leq 1 + \alpha - \frac{1}{2}\Delta$,

(4) $\psi(x)$ has a Fourier series expansion of the form

$$\psi(x) = \beta - \alpha + \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} A(\nu)e(\nu x),$$

where

$$(3.4) \quad |A(\nu)| \ll \min \left(\frac{1}{\nu}, \beta - \alpha, \frac{1}{\nu^2 \Delta} \right).$$

We note that we could have used Vaaler polynomials [22], however, we do not gain anything by doing so as the estimates we get are already best possible. Setting

$$(3.5) \quad \begin{aligned} \alpha_- &= \sum_{\lambda=1}^{\ell} d_{\lambda} q^{-\lambda} + (2\delta)^{-1}, & \beta_- &= \sum_{\lambda=1}^{\ell} d_{\lambda} q^{-\lambda} + q^{-\ell} - (2\delta)^{-1}, \\ \delta &= P^{-\gamma}, \\ \alpha_+ &= \sum_{\lambda=1}^{\ell} d_{\lambda} q^{-\lambda} - (2\delta)^{-1}, & \beta_+ &= \sum_{\lambda=1}^{\ell} d_{\lambda} q^{-\lambda} + q^{-\ell} + (2\delta)^{-1}. \end{aligned}$$

and an application of Lemma 3.1 with $(\alpha, \beta, \delta) = (\alpha_-, \beta_-, \delta)$ and $(\alpha, \beta, \delta) = (\alpha_+, \beta_+, \delta)$, respectively, provides us with two functions \mathcal{I}_- and \mathcal{I}_+ . By our choice of $(\alpha_{\pm}, \beta_{\pm}, \delta)$ it is immediate that

$$(3.6) \quad \mathcal{I}_-(t) \leq \mathcal{I}(t) \leq \mathcal{I}_+(t) \quad (t \in \mathbb{R}).$$

Lemma 3.1 also implies that these two functions have Fourier expansions

$$(3.7) \quad \mathcal{I}_{\pm}(t) = q^{-\ell} \pm P^{-\gamma} + \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} A_{\pm}(\nu)e(\nu t)$$

satisfying

$$|A_{\pm}(\nu)| \ll \min(|\nu|^{-1}, P^{\gamma} |\nu|^{-2}).$$

In a next step we want to replace \mathcal{I} by \mathcal{I}_+ in (3.3). For this purpose we observe, using (3.6), and (3.7) that

$$|\mathcal{I}(t) - q^{-\ell}| \ll P^{-\gamma} + \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} A_{\pm}(\nu)e(\nu t).$$

Thus setting $t = q^{-j}f(p)$ and summing over $p \leq P$ yields

$$(3.8) \quad \left| \sum_{p \leq P} \mathcal{I}(q^{-j}f(p)) - \frac{\pi(P)}{q^\ell} \right| \ll \pi(P)P^{-\gamma} + \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} A_{\pm}(\nu) \sum_{p \leq P} e\left(\frac{\nu}{q^j}f(p)\right).$$

Now we consider the coefficients $A_{\pm}(\nu)$. Noting (3.4) one observes that

$$A_{\pm}(\nu) \ll \begin{cases} \nu^{-1}, & \text{for } |\nu| \leq P^\gamma; \\ P^\gamma \nu^{-2}, & \text{for } |\nu| > P^\gamma. \end{cases}$$

Estimating all summands with $|\nu| > P^\gamma$ trivially we get

$$\sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} A_{\pm}(\nu) e\left(\frac{\nu}{q^j}f(p)\right) \ll \sum_{\nu=1}^{P^\gamma} \nu^{-1} e\left(\frac{\nu}{q^j}f(p)\right) + P^{-\gamma}.$$

Using this in (3.8) yields

$$\left| \sum_{p \leq P} \mathcal{I}(q^{-j}f(p)) - \frac{\pi(P)}{q^\ell} \right| \ll \pi(P)P^{-\gamma} + \sum_{\nu=1}^{P^\gamma} \nu^{-1} S(P, j, \nu),$$

where we have set

$$(3.9) \quad S(P, j, \nu) := \sum_{p \leq P} e\left(\frac{\nu}{q^j}f(p)\right).$$

4. EXPONENTIAL SUM ESTIMATES

In the present section we will focus on the estimation of the sum $S(P, j, \nu)$ for different ranges of j . Since j describes the position within the q -ary expansion of $f(p)$ we will call these ranges the “most significant digits”, the “least significant digits” and the “digits in the middle”, respectively.

Now, if $\theta_r > k \geq 0$, *i.e.* the leading coefficient of f origins from the pseudo polynomial part g , then we consider the two ranges

$$1 \leq q^j \leq P^{\theta_r - \rho} \quad \text{and} \quad P^{\theta_r - \rho} < q^j \leq P^{\theta_r}.$$

For the first one we will apply Proposition 4.3 and for the second one Proposition 4.1.

On the other hand, if $k > \theta_r > 0$, meaning that the leading coefficient of f origins from the polynomial part h , then we have an additional part. In particular, in this case we will consider the three ranges

$$1 \leq q^j \leq P^{\theta_r - \rho}, \quad P^{\theta_r - \rho} < q^j \leq P^{k-1+\rho}, \quad \text{and} \quad P^{k-1+\rho} < q^j \leq P^k.$$

We will, similar to above, treat the first and last range by Proposition 4.3 and Proposition 4.1, respectively. For the middle range we will apply Proposition 4.7. Since $2\rho < \theta_r$, we note that the middle range is empty if $k = 1$.

Since the size of j represents the position of the digit in the expansion (cf. (3.2)), we will deal in the following subsection with the “most significant digits”, the “least significant digits” and the “digits in the middle”, respectively.

4.1. Most significant digits. We start our series of estimates for the exponential sum $S(P, j, \nu)$ for j being in the highest range. In particular, we want to show the following

Proposition 4.1. *Suppose that for some $k \geq 1$ we have $|f^{(k)}(x)| \geq \Lambda$ for any x on $[a, b]$ with $\Lambda > 0$. Then*

$$S(P, j, \nu) \ll \frac{1}{\log P} \Lambda^{-\frac{1}{k}} + \frac{P}{(\log P)^G}.$$

The main idea of the proof is to use Riemann-Stieltjes integration together with

Lemma 4.2 ([9, Lemma 8.10]). *Let $F: [a, b] \rightarrow \mathbb{R}$ and suppose that for some $k \geq 1$ we have $|F^{(k)}(x)| \geq \Lambda$ for any x on $[a, b]$ with $\Lambda > 0$. Then*

$$\left| \int_a^b e(F(x)) dx \right| \leq k 2^k \Lambda^{-1/k}.$$

Proof of Proposition 4.1. We rewrite the sum into a Riemann-Stieltjes integral:

$$S(P, j, \nu) = \sum_{p \leq P} e\left(\frac{\nu}{q^j} f(p)\right) = \int_2^P e\left(\frac{\nu}{q^j} f(t)\right) d\pi(t) + \mathcal{O}(1).$$

Then we apply the prime number theorem in the form (2.2) to gain the usual integral back. Thus

$$S(P, j, \nu) = \int_{P(\log P)^{-G}}^P e\left(\frac{\nu}{q^j} f(t)\right) \frac{dt}{\log t} + \mathcal{O}\left(\frac{P}{(\log P)^G}\right).$$

Now we use the second mean-value theorem to get

$$(4.1) \quad S(P, j, \nu) \ll \frac{1}{\log P} \sup_{\xi} \left| \int_{P(\log P)^{-G}}^{\xi} e\left(\frac{\nu}{q^j} f(t)\right) dt \right| + \frac{P}{(\log P)^G}.$$

Finally an application of Lemma 4.2 proves the lemma. \square

4.2. Least significant digits. Now we turn our attention to the lowest range of j . In particular, the goal is the proof of the following

Proposition 4.3. *Let P and ρ be positive reals and f be a pseudo-polynomial as in (3.1). If j is such that*

$$(4.2) \quad 1 \leq q^j \leq P^{\theta_r - \rho}$$

holds, then for $1 \leq \nu \leq P^\gamma$ there exists $\eta > 0$ (depending only on f and ρ) such that

$$S(P, j, \nu) = (\log P)^8 P^{1-\eta}.$$

Before we launch into the proof we collect some tools that will be necessary in the sequel. A standard idea for estimating exponential sums over the primes is to rewrite them into ordinary exponential sums over the integers having von Mangoldt's function as weights and then to apply Vaughan's identity. We denote by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some prime } p \text{ and an integer } k \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

von Mangoldt's function. For the rewriting process we use the following

Lemma 4.4. *Let g be a function such that $|g(n)| \leq 1$ for all integers n . Then*

$$\left| \sum_{p \leq P} g(p) \right| \ll \frac{1}{\log P} \max_{t \leq P} \left| \sum_{n \leq t} \Lambda(n) g(n) \right| + \mathcal{O}(\sqrt{P}).$$

Proof. This is Lemma 11 of [15]. However, the proof is short and we need some piece later.

We start with a summation by parts yielding

$$\sum_{p \leq P} g(p) = \frac{1}{\log P} \sum_{p \leq x} \log(p) g(p) + \int_2^P \left(\sum_{p \leq t} \log(p) g(p) \right) \frac{dt}{t(\log t)^2}.$$

Now we cut the integral at \sqrt{P} and use Chebyshev's inequality (cf. [21, Théorème 1.3]) in the form $\sum_{p \leq t} \log(p) \leq \log(t)\pi(t) \ll t$ for the lower part. Thus

$$\begin{aligned} \left| \sum_{p \leq P} g(p) \right| &\leq \left(\frac{1}{\log P} + \int_{\sqrt{P}}^P \frac{dt}{t(\log t)^2} \right) \max_{\sqrt{P} < t \leq P} \left| \sum_{p \leq t} \log(p) g(p) \right| + \mathcal{O}(\sqrt{P}) \\ &= \frac{2}{\log P} \max_{\sqrt{P} < t \leq P} \left| \sum_{p \leq t} \log(p) g(p) \right| + \mathcal{O}(\sqrt{P}). \end{aligned}$$

Finally we again use Chebyshev's inequality $\pi(t) \ll t/\log(t)$ to obtain

$$(4.3) \quad \left| \sum_{n \leq t} \Lambda(n)g(n) - \sum_{p \leq t} \log(p)g(p) \right| \leq \sum_{p \leq \sqrt{t}} \log(p) \sum_{a=2}^{\lfloor \frac{\log(t)}{\log(p)} \rfloor} 1 \leq \pi(\sqrt{t}) \log(t) \ll \sqrt{t}.$$

□

In the next step we use Vaughan's identity to subdivide this weighted exponential sum into several sums of Type I and II.

Lemma 4.5 ([1, Lemma 2.3]). *Assume $F(x)$ to be any function defined on the real line, supported on $[P/2, P]$ and bounded by F_0 . Let further U, V, Z be any parameters satisfying $3 \leq U < V < Z < P$, $Z \geq 4U^2$, $P \geq 64Z^2U$, $V^3 \geq 32P$ and $Z - \frac{1}{2} \in \mathbb{N}$. Then*

$$\left| \sum_{P/2 < n \leq P} \Lambda(n)F(n) \right| \ll K \log P + F_0 + L(\log P)^8,$$

where K and L are defined by

$$K = \max_M \sum_{m=1}^{\infty} d_3(m) \left| \sum_{Z < n \leq M} F(mn) \right|,$$

$$L = \sup \sum_{m=1}^{\infty} d_4(m) \left| \sum_{U < n < V} b(n)F(mn) \right|,$$

where the supremum is taken over all arithmetic functions $b(n)$ satisfying $|b(n)| \leq d_3(n)$.

After subdividing the weighted exponential sum with Vaughan's identity we will use the following lemma in order to estimate the occurring exponential sums.

Lemma 4.6 ([1, Lemma 2.5]). *Let $X, k, q \in \mathbb{N}$ with $k, q \geq 0$ and set $K = 2^k$ and $Q = 2^q$. Let $h(x)$ be a polynomial of degree k with real coefficients. Let $g(x)$ be a real $(q + k + 2)$ times continuously differentiable function on $[X/2, X]$ such that $|f^{(r)}(x)| \asymp FX^{-r}$ ($r = 1, \dots, q + k + 2$). Then, if $F = o(X^{q+2})$ for F and X large enough, we have*

$$\left| \sum_{X/2 < x \leq X} e(g(x) + h(x)) \right| \ll X^{1-\frac{1}{K}} + X \left(\frac{\log^k X}{F} \right)^{\frac{1}{K}} + X \left(\frac{F}{X^{q+2}} \right)^{\frac{1}{(4KQ-2K)}}.$$

Now we have the necessary tools to state the

Proof of Proposition 4.3. An application of Lemma 4.4 yields

$$S(P, j, \nu) \ll \frac{1}{\log P} \max \left| \sum_{n \leq P} \Lambda(n) e \left(\frac{\nu}{q^j} (g(n) + h(n)) \right) \right| + P^{\frac{1}{2}}.$$

We split the inner sum into $\leq \log P$ sub sums of the form

$$\left| \sum_{X < n \leq 2X} \Lambda(n) e \left(\frac{\nu}{q^j} (g(n) + h(n)) \right) \right|$$

with $2X \leq P$ and let S be a typical one of them. We may assume that $X \geq P^{1-\rho}$.

Using Vaughan's identity (Lemma 4.5) with $U = \frac{1}{4}X^{1/5}$, $V = 4X^{1/3}$ and Z the unique number in $\frac{1}{2} + \mathbb{N}$, which is closest to $\frac{1}{4}X^{2/5}$, we obtain

$$(4.4) \quad S \ll 1 + (\log X)S_1 + (\log X)^8 S_2,$$

where

$$\begin{aligned} S_1 &= \sum_{x < \frac{2X}{Z}} d_3(x) \sum_{y > Z, \frac{X}{x} < y < \frac{2X}{x}} e \left(\frac{\nu}{q^j} (g(xy) + h(xy)) \right) \\ S_2 &= \sum_{\frac{X}{V} < x \leq \frac{2X}{U}} d_4(x) \sum_{U < y < V, \frac{X}{x} < y \leq \frac{2X}{x}} b(y) e \left(\frac{\nu}{q^j} (g(xy) + h(xy)) \right) \end{aligned}$$

We start with the estimation of S_1 . Since $d_3(x) \ll x^\varepsilon$ we have for

$$|S_1| \ll X^\varepsilon \sum_{x \leq \frac{2X}{Z}} \left| \sum_{\substack{\frac{X}{x} < y < \frac{2X}{x} \\ y > Z}} e \left(\frac{\nu}{q^j} (g(xy) + h(xy)) \right) \right|.$$

For estimating the inner sum we fix x and denote $Y = \frac{X}{x}$. Since $\theta_r \notin \mathbb{Z}$ and $\theta_r > k \geq 0$, we have that

$$\left| \frac{\partial^\ell g(xy)}{\partial y^\ell} \right| \asymp X^{\theta_r} Y^{-\ell}.$$

Now on the one hand, since $q^j \leq P^{\theta_r - \rho}$, we have $\nu q^{-j} X^{\theta_r} \gg X^\rho$. On the other hand for $\ell \geq 5(\lfloor \theta_r \rfloor + 1)$ we get

$$\frac{\nu}{q^j} X^{\theta_r} Y^{-\ell} \leq P^\gamma X^{\theta_r - \frac{2}{5}\ell} \ll X^{-\frac{1}{2}}.$$

Thus an application of Lemma 4.6 yields the following estimate:

$$\begin{aligned}
 (4.5) \quad |S_1| &\ll X^\varepsilon \sum_{x \leq 2X/Z} Y \left[Y^{-\frac{1}{K}} + (\log Y)^k X^{-\frac{\rho}{K}} + X^{-\frac{1}{2} \frac{1}{4K \cdot 8L^5 - 2K}} \right] \\
 &\ll X^{1+\varepsilon} (\log X) \left(X^{-\rho} + X^{-\frac{1}{64L^5 - 4}} \right)^{\frac{1}{K}},
 \end{aligned}$$

where we have used that $\frac{k}{K} < 1$ and $\rho < \frac{1}{3}$.

For the second sum S_2 we start by splitting the interval $(\frac{X}{V}, \frac{2X}{U}]$ into $\leq \log X$ subintervals of the form $(X_1, 2X_1]$. Thus

$$|S_2| \leq (\log X) X^\varepsilon \sum_{X_1 < x \leq 2X_1} \left| \sum_{\substack{U < y < V \\ \frac{X}{x} < y \leq \frac{2X}{x}}} b(y) e \left(\frac{\nu}{q^j} (g(xy) + h(xy)) \right) \right|$$

Now an application of Cauchy's inequality together with $|b(y)| \ll X^\varepsilon$ yields

$$\begin{aligned}
 |S_2|^2 &\leq (\log X)^2 X^{2\varepsilon} X_1 \sum_{X_1 < x \leq 2X_1} \left| \sum_{\substack{U < y < V \\ \frac{X}{x} < y \leq \frac{2X}{x}}} b(y) e \left(\frac{\nu}{q^j} (g(xy) + h(xy)) \right) \right|^2 \\
 &\ll (\log X)^2 X^{4\varepsilon} X_1 \\
 &\quad \times \left(X_1 \frac{X}{X_1} + \left| \sum_{X_1 < x \leq 2X_1} \sum_{A < y_1 < y_2 \leq B} e \left(\frac{\nu}{q^j} (g(xy_1) - g(xy_2) + h(xy_1) - h(xy_2)) \right) \right| \right)
 \end{aligned}$$

where $A = \max\{U, \frac{X}{x}\}$ and $B = \min\{U, \frac{2X}{x}\}$. Changing the order of summation, we get

$$\begin{aligned}
 |S_2|^2 &\ll (\log X)^2 X^{4\varepsilon} X_1 \\
 &\times \left(X + \sum_{A < y_1 < y_2 \leq B} \left| \sum_{X_1 < x \leq 2X_1} e \left(\frac{\nu}{q^j} (g(xy_1) - g(xy_2) + h(xy_1) - h(xy_2)) \right) \right| \right)
 \end{aligned}$$

As above we want to apply Lemma 4.6. To this end we fix y_1 and $y_2 \neq y_1$. Similarly to above we get that

$$\left| \frac{\partial^\ell (g(xy_1) - g(xy_2) + h(xy_1) - h(xy_2))}{\partial x^\ell} \right| \asymp \frac{|y_1 - y_2|}{y_1} X^{\theta_r} X_1^{-\ell}.$$

Now, on the one hand we have $\frac{\nu}{q^j} \frac{|y_1 - y_2|}{y_1} X^{\theta_r} \gg X^\rho$ and on the other hand

$$\frac{\nu}{q^j} \frac{|y_1 - y_2|}{y_1} X^{\theta_r} X_1^{-\ell} \ll X^{\gamma + \theta_r} \left(\frac{X}{V} \right)^{-\ell} \ll X^{\gamma + \theta_r - \frac{2}{3}\ell} \ll X^{-\frac{1}{2}}$$

if $\ell \geq 2[\theta_r] + 3$. Thus again an application of Lemma 4.6 yields

(4.6)

$$\begin{aligned} |S_2|^2 &\ll (\log X)^2 X^{4\varepsilon} X_1 \left(X + \sum_{A < y_1 < y_2 \leq B} X_1 \left(X_1^{-\frac{1}{K}} + X^{-\frac{\rho}{K}} + X^{-\frac{1}{2} \frac{1}{4K \cdot 2L^2 - 2K}} \right) \right) \\ &\ll (\log X)^2 X^{4\varepsilon} \left(X^{\frac{5}{3}} + X^{2 - \frac{\rho}{K}} + X^{2 - \frac{1}{16KL^2 - 4K}} \right). \end{aligned}$$

Plugging the two estimates (4.5) and (4.6) into (4.4) proves the proposition. \square

4.3. The digits in the middle. Now we are getting more involved in order to consider those j leading to a position between θ_r and k . These sums correspond to the “digits in the middle” in the proof of Theorem 2.1. We want to prove the following

Proposition 4.7. *Let P and ρ be positive reals and f be a pseudo-polynomial as in (3.1). If $2\rho < \theta_r < k$ and j is such that*

$$(4.7) \quad P^{\theta_r - \rho} < q^j \leq P^{k-1+\rho}$$

holds, then for $1 \leq \nu \leq P^\gamma$ we have

$$S(P, j, \nu) = \sum_{p \leq P} e \left(\frac{\nu f(p)}{q^j} \right) \ll P^{1 - \frac{\rho}{4k}}.$$

The main idea in this range is to use that the dominant part of f comes from the polynomial h . Therefore after getting rid of the function g we will estimate the sum over the polynomial by the following

Lemma 4.8. *Let $h \in \mathbb{R}[X]$ be a polynomial of degree $k \geq 2$. Suppose α is the leading coefficient of h and that there are integers a, q such that*

$$|q\alpha - a| < \frac{1}{q} \quad \text{with} \quad (a, q) = 1.$$

Then we have for any $\varepsilon > 0$ and $H \leq X$

$$\sum_{X < p \leq X+H} \log(p) e(h(p)) \ll H^{1+\varepsilon} \left(\frac{1}{q} + \frac{1}{H^{\frac{1}{2}}} + \frac{q}{H^k} \right)^{4^{1-k}}.$$

Proof. This is a slight variant of [8, Theorem 1], where we sum over an interval of the form $]X, X+H]$ instead of one of the form $]0, X]$. \square

Now we have enough tools to state the

Proof of Proposition 4.7. As in the Proof of Proposition 4.3 we start by an application of Lemma 4.4 yielding

$$S(P, j, \nu) \ll \frac{1}{\log P} \max \left| \sum_{n \leq P} \Lambda(n) e \left(\frac{\nu}{q^j} (g(n) + h(n)) \right) \right| + P^{\frac{1}{2}}.$$

We split the inner sum into $\leq \log P$ sub sums of the form

$$S := \sum_{X < n \leq X+H} \Lambda(n) e \left(\frac{\nu}{q^j} (g(n) + h(n)) \right)$$

with $P^{1-2\rho} \leq X \leq P$ and

$$H = \min \left(P^{1-\theta_r} |\nu|^{-1} q^j, X \right).$$

Now we want to separate the function parts g and h . Therefore we define two functions T and φ by

$$T(x) = \sum_{X < n \leq X+x} \Lambda(n) e \left(\frac{\nu}{q^j} h(n) \right) \quad \text{and} \quad \varphi(x) := e \left(\frac{\nu}{q^j} g(X+x) \right)$$

Then an application of summation by parts yields

(4.8)

$$\begin{aligned} \sum_{X < n \leq X+H} \Lambda(n) e \left(\frac{\nu}{q^j} (g(n) + h(n)) \right) &= \sum_{n=1}^H \varphi(n) (T(n) - T(n-1)) \\ &= \sum_{n=1}^H T(n) (\varphi(n) - \varphi(n+1)) + \varphi(H-1) T(H) \\ &\ll |T(H)| + \sum_{n=1}^{H-1} |\varphi(n) - \varphi(n+1)| |T(n)| \end{aligned}$$

Let α_k be the leading coefficient of P . Then by Diophantine approximation there always exists a rational a/b with $b > 0$, $(a, b) = 1$,

$$1 \leq b \leq H^{k-\rho} \quad \text{and} \quad \left| \frac{\nu \alpha_k}{q^j} - \frac{a}{b} \right| \leq \frac{H^{\rho-k}}{b}.$$

We distinguish three cases according to the size of b .

Case 1. $H^\rho < b$. In this case we may apply Lemma 4.8 together with (4.3) to get

$$T(h) \ll H^{1 - \frac{\rho}{4^{k-1}} + \varepsilon}.$$

Case 2. $2 \leq b < H^\rho$. In this case we get that

$$\left| \frac{\nu\alpha_k}{q^j} \right| \geq \left| \frac{a}{b} \right| - \frac{1}{b^2} \geq \frac{1}{2b} \geq \frac{1}{2} H^{-\rho} \geq \frac{1}{2} P^{-\rho}.$$

Since $2\rho < \theta_r$, this contradicts our lower bound $q^j \geq P^{\theta_r - \rho}$.

Case 3. $b = 1$. This case requires a further distinction according to whether $a = 0$ or not.

Case 3.1. $\left| \frac{\nu\alpha_k}{q^j} \right| \geq \frac{1}{2}$. It follows that

$$q^j \leq 2 |\nu\alpha_k|$$

again contradicting our lower bound $q^j \geq P^{\theta_r - \rho}$.

Case 3.2. $\left| \frac{\nu\alpha_k}{q^j} \right| < \frac{1}{2}$. This implies that $a = 0$ which yields

$$(4.9) \quad q^j \geq |\nu\alpha_k| H^{k-\rho}.$$

We distinguish two further cases according to whether $P^{1-\theta_r} |\nu|^{-1} q^j \leq X$ or not.

Case 3.2.1 $P^{1-\theta_r} |\nu|^{-1} q^j \leq X$. This implies that $q^j \leq P^{\theta_r} |\nu|$ and

$$H = P^{1-\theta_r} |\nu|^{-1} q^j \geq P^{1-\rho} |\nu|^{-1} \geq P^{1-2\rho}.$$

Plugging these estimates into (4.9) gives

$$P^{\theta_r} \geq |\alpha_k| P^{(1-2\rho)(k-\rho)}.$$

However, since $4(k+1)\rho < 1$, we have

$$(1-2\rho)(k-\rho) > k-1+2\rho \geq \theta_r$$

yielding a contradiction.

Case 3.2.2 $P^{1-\theta_r} |\nu|^{-1} q^j > X$. Then $H = X \geq P^{1-2\rho}$ and (4.9) becomes

$$P^{k-1+\rho} \geq |\nu\alpha_k| P^{(1-2\rho)(k-\rho)}$$

yielding a similar contradiction as in **Case 3.2.1**.

Therefore **Case 1** is the only possible and we may always apply Lemma 4.8 together with (4.3). Plugging this into (4.8) yields

$$\sum_{X < n \leq X+H} \Lambda(n) e \left(\frac{\nu}{q^j} (g(n) + h(n)) \right) \ll H^{1-\frac{\rho}{4k-1}+\varepsilon} \left(1 + \sum_{X < n \leq X+H} |\varphi(n) - \varphi(n+1)| \right)$$

Now by our choice of H together with an application of the mean value theorem we have that

$$\sum_{X \leq n \leq X+H} |\varphi(n) - \varphi(n+1)| \ll H \frac{\nu}{q^j} P^{\theta-1} \ll 1.$$

Thus

$$\sum_{X \leq n \leq X+H} \Lambda(n) e\left(\frac{\nu}{q^j}(g(n) + h(n))\right) \ll H^{1-\frac{\rho}{4^{k-1}}+\varepsilon}.$$

□

5. PROOF OF THEOREM 2.1, PART II

Now we use all the tools from the section above in order to estimate

$$(5.1) \quad \sum_{j=\ell}^J \left| \sum_{p \leq P} \mathcal{I}(q^{-j} f(p)) - \frac{\pi(P)}{q^\ell} \right| \ll \pi(P) H^{-1} J + \sum_{\nu=1}^H \nu^{-1} \sum_{j=\ell}^J S(P, j, \nu).$$

As indicated in the section above, we split the sum over j into two or three parts according to whether $\theta_r > k$ or not. In any case an application of Proposition 4.3 yields for the least significant digits that

$$(5.2) \quad \sum_{1 \leq \nu \leq P^\gamma} \nu^{-1} \sum_{1 \leq q^j \leq P^{\theta_r - \rho}} S(P, j, \nu) \ll (\log P)^9 J P^{1-\eta}.$$

Now let us suppose that $\theta_r > k$. Then an application of Proposition 4.1 yields

$$(5.3) \quad \begin{aligned} & \sum_{1 \leq \nu \leq P^\gamma} \nu^{-1} \sum_{P^{\theta_r - \rho} < q^j \leq P^{\theta_r}} S(P, j, \nu) \\ & \ll \sum_{1 \leq \nu \leq P^\gamma} \nu^{-1} \sum_{P^{\theta_r - \rho} < q^j \leq P^{\theta_r}} \frac{1}{\log P} \left(\frac{\nu}{q^j} \right)^{-\frac{1}{\lceil \theta_r \rceil}} + \frac{P}{(\log P)^{G-2}} \\ & \ll \frac{P}{\log P}. \end{aligned}$$

Plugging the estimates (5.2) and (5.3) into (5.1) we get that

$$\sum_{j=\ell}^J \left| \sum_{p \leq P} \mathcal{I}(q^{-j} f(p)) - \frac{\pi(P)}{q^\ell} \right| \ll \frac{P}{\log P},$$

which together with (3.3) proves Theorem 2.1 in the case that $\theta_r > k$.

On the other side if $\theta_r < k$, then we consider the two ranges

$$P^{\theta_r - \rho} < q^j \leq P^{k-1+\rho} \quad \text{and} \quad P^{k-1+\rho} < q^j \leq P^k.$$

For the “digits in the middle” an application of Proposition 4.7 yields

$$(5.4) \quad \sum_{1 \leq \nu \leq P^\gamma} \nu^{-1} \sum_{P^{\theta_r - \rho} < q^j \leq P^{k-1+\rho}} S(P, j, \nu) \ll \sum_{1 \leq \nu \leq P^\gamma} \nu^{-1} \sum_{P^{\theta_r - \rho} < q^j \leq P^{k-1+\rho}} P^{1 - \frac{\rho}{4k}} \\ \ll (\log P) J P^{1 - \frac{\rho}{4k}}.$$

Finally we consider the most significant digits. By an application of Proposition 4.1 we have

$$(5.5) \quad \sum_{1 \leq \nu \leq P^\gamma} \nu^{-1} \sum_{P^{k-1+\rho} < q^j \leq P^k} S(P, j, \nu) \\ \ll \sum_{1 \leq \nu \leq P^\gamma} \nu^{-1} \sum_{P^{k-1+\rho} < q^j \leq P^k} \frac{1}{\log P} \left(\frac{\nu}{q^j} \right)^{-\frac{1}{k}} + \frac{P}{(\log P)^{G-2}} \\ \ll \frac{P}{\log P}.$$

Plugging the estimates (5.2), (5.4) and (5.5) into (5.1) we get that

$$\sum_{j=\ell}^J \left| \sum_{p \leq P} \mathcal{I}(q^{-j} f(p)) - \frac{\pi(P)}{q^\ell} \right| \ll \frac{P}{\log P},$$

which together with (3.3) proves Theorem 2.1 in the case that $\theta_r < k$.

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